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TRANSLATIONAL ADDITION THEOREMS FOR SPHERICAL VECTOR WAVE FUNCTIONS

Orval R. Cruzan

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FOR THE COMMANDER:
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B. M. Horton
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ABSTRACT

Translational addition theorems for spherical vector wave functions have been derived in a reduced form. The reduction has been accomplished by the use of formulas relating the coefficients that arise in expansion of the product of two associated Legendre functions. These addition theorems should be useful in those cases in which spherical vector wave functions are used where the distances of bodies and sources are separated by the order of a few wavelengths.

1. INTRODUCTION

When electromagnetic waves interact with spherical bodies, it becomes desirable in many problems to expand the fields in terms of spherical vector wave functions. If several spherical bodies are involved and if the radii of curvature of the waves are appreciable at the positions of the spherical bodies, it is helpful to have addition theorems for the wave functions.

Such addition theorems have been obtained by Stein (ref 1).* As given for the case of coordinate translation they are not in the most desirable form, in that the coefficients involved consist of several terms each. Although these coefficients were not reduced, it was stated that certain recursion formulas might be useful.

It turns out that in addition to two recursion formulas (C-11) and (C-12), which were considered most useful, four other formulas are required, two of which are of the recursion form; the other two have a quasi-recurrent form, as shown in equation (C-7) and (C-8). These formulas have been obtained, and the coefficients reduced to one term each. Moreover, the derivation of the addition theorems has been made in a straightforward and rigorous way, by making use of the orthogonal properties of the angular functions and of the vector wave functions.

For the case of coordinate translation the following addition theorems have been obtained. The vector wave functions on the left-hand side of the equations refer to the original set of coordinates, while those on the right-hand side refer to the translated coordinates. Accordingly,

$$\text{Theorem I:}\quad \bar{m}_{mn} = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} \left(A_{\mu\nu}^{mn}\, \bar{m}_{\mu\nu} + B_{\mu\nu}^{mn}\, \bar{n}_{\mu\nu} \right)$$

$$\text{Theorem II:}\quad \bar{n}_{mn} = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} \left(A_{\mu\nu}^{mn}\, \bar{n}_{\mu\nu} + B_{\mu\nu}^{mn}\, \bar{m}_{\mu\nu} \right),$$

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* A list of references appears on page 30.

where

$$A_{\mu\nu}^{mn} = (-1)^{\mu} \sum_p a(m,n|-\mu,\nu|p)\ a(n,\nu,p)\ z_p(ka)\ P_p^{m-\mu}(\cos\theta_o)\ \exp[\,i(m-\mu)\phi_o\,]\,,$$

$$B_{\mu\nu}^{mn} = (-1)^{\mu+1} \sum_p a(m,n|-\mu,\nu|p,p-1)\ b(n,\nu,p)\ z_p(ka)\ P_p^{m-\mu}(\cos\theta_o)\ \exp[\,i(m-\mu)\phi_o\,]\,,$$

$$a(n,\nu,p) = i^{\nu+p-n}[2\nu(\nu+1)(2\nu+1)+(\nu+1)(n-\nu+p+1)(n+\nu-p)-\nu(\nu-n+p+1)(n+\nu+p+2)]/[2\nu(\nu+1)]\,,$$

and

$$(n,\nu,p) = i^{\nu+p-n}[(n+\nu+p+1)(\nu-n+p)(n-\nu+p)(n+\nu-p+1)]^{\frac{1}{2}}(2\nu+1)/[2\nu(\nu+1)]\ .$$

The other coefficient factors are given explicitly in the text.

2. SPHERICAL VECTOR WAVE FUNCTIONS (ref 2)

Divergenceless solutions of the vector wave equation,

$$\nabla\nabla\cdot\bar{C} - \nabla\times\nabla\times\bar{C} + k^2\bar{C} = 0, \qquad (1)$$

are the two vector wave functions $\bar{M}$ and $\bar{N}$. The relations between these two are

$$k\bar{N} = \nabla\times\bar{M} \qquad (2)$$

$$k\bar{M} = \nabla\times\bar{N}.$$

In spherical coordinates, $\bar{M}$ is given by the formula

$$\bar{M} = \nabla\times\bar{r}\psi, \qquad (3)$$

where $\bar{r}$ is a radial vector, and ψ is a solution of the scalar Helmholtz equation,

$$\nabla^2\psi + k^2\psi = 0. \qquad (4)$$

By means of vector identities, formula (3) can be put in the alternative form,

$$\bar{M} = \nabla\psi\times\bar{r}, \qquad (5)$$

since $\nabla \times \bar{r} = 0$. Omitting the time factor exp $(-i\omega t)$, the explicit expressions for $\bar{M}$ and $\bar{N}$ respectively are (ref 3):

$$\bar{m}_{mn} = \frac{im}{\sin\theta} z_n(kr) P_n^m(\cos\theta) \exp(im\phi)\, i_\theta - z_n(kr) \frac{\partial P_n^m}{\partial\theta}(\cos\theta) \exp(im\phi)\, i_\phi$$

and

$$\bar{n}_{mn} = \frac{n(n+1)}{kr} z_n(kr) P_n^m(\cos\theta) \exp(im\phi)\, i_r + \frac{1}{kr}\frac{\partial}{\partial r}\left[r\, z_n(kr)\right] \frac{\partial P_n^m}{\partial\theta}(\cos\theta) \exp(im\phi)\, i_\theta + \frac{im}{kr\sin\theta}\frac{\partial}{\partial r}\left[r\, z_n(kr)\right] P_n^m(\cos\theta) \exp(im\phi)\, i_\phi , \quad (6)$$

where $z_n(kr)$ stands for any of the radial functions.

3. VECTOR WAVE FUNCTIONS UNDER COORDINATE TRANSLATION

If the translation as illustrated in figure 1 is made, then

$$\bar{r} = \bar{a} + \bar{r}_1 . \quad (7)$$

With this value of $\bar{r}$, we have

$$\bar{M} = \nabla\psi \times \bar{a} + \nabla\psi \times \bar{r}_1 . \quad (8)$$

Since the gradient of a scalar quantity is invariant to a transformation of the coordinate system, then we may regard $\nabla\psi$ as being expressed in terms of the coordinates of the second system. As can be determined from figure 1,

$$\bar{a} = a(i_x \sin\theta_o \cos\phi_o + i_y \sin\theta_o \sin\phi_o + i_z \cos\theta_o) . \quad (9)$$

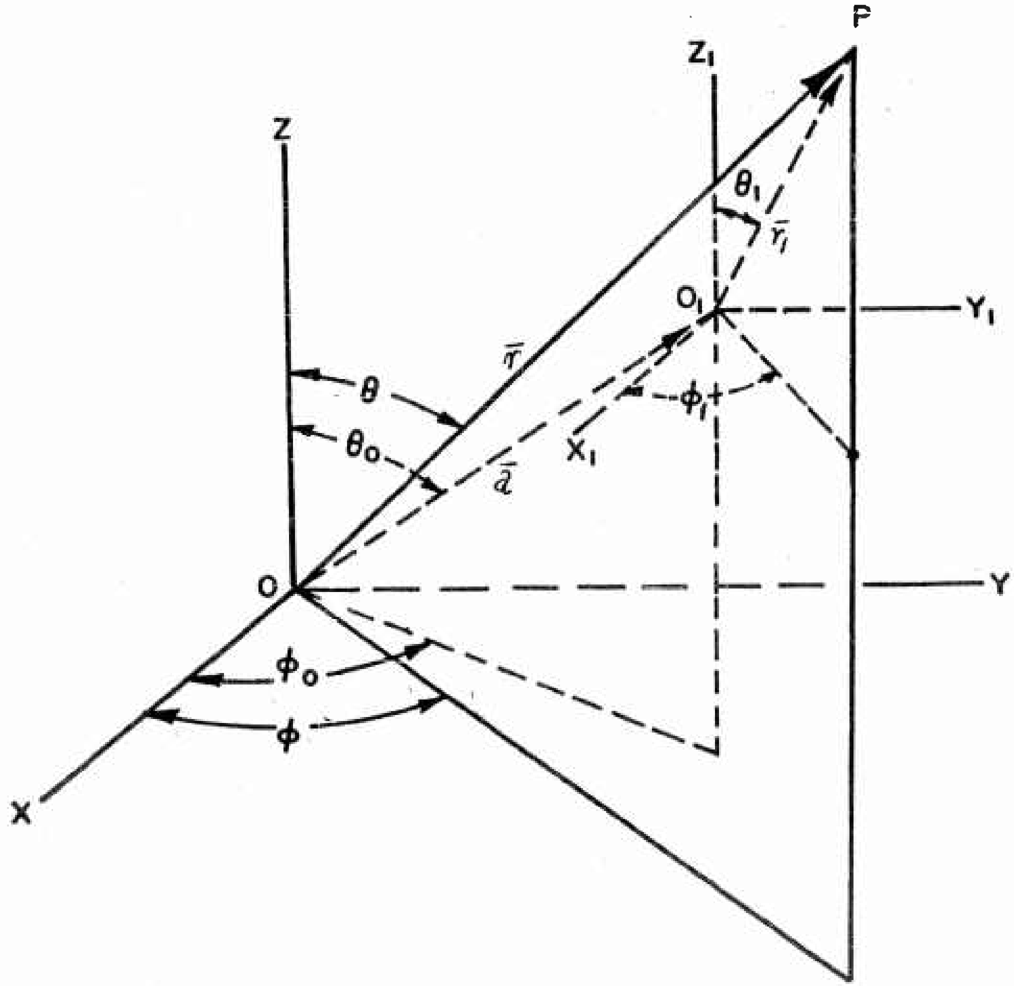

Figure 1. Coordinate translation

Consequently, we have

$$\overline{M} = a(\sin\theta_o \cos\phi_o \nabla\psi \times i_x + \sin\theta_o \sin\phi_o \nabla\psi \times i_y + \cos\theta_o \nabla\psi \times i_z)$$

$$+ \nabla\psi \times \overline{r}_1 \ . \tag{10}$$

The unit vectors (i_x, i_y, i_z) in terms of the unit vectors in the second spherical coördinate system are

$$i_x = i_{r_1} \sin\theta_1 \cos\phi_1 + i_{\theta_1} \cos\theta_1 \cos\phi_1 - i_{\phi_1} \sin\phi_1$$

$$i_y = i_{r_1} \sin\theta_1 \sin\phi_1 + i_{\theta_1} \cos\theta_1 \sin\phi_1 + i_{\phi_1} \cos\phi_1$$

$$i_z = i_{r_1} \cos\theta_1 - i_{\theta_1} \sin\theta_1 \ .$$

If in equation (B-1) we put

$$A(\mu,\nu) = (-1)^{\mu}\, i^{\nu-n} \sum_p i^p\, a(m,n|-\mu,\nu|p) z_p(ka)\, P_p^{m-\mu}(\cos\theta_o) \exp\left[i(m-\mu)\phi_o\right] , \tag{12}$$

where the $a(m,n|-\mu\nu|p)$'s ~~are~~ obtained in appendix A, then

$$\psi = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} A(\mu,\nu)\, j_\nu(kr_1)\, P_\nu^{\mu}(\cos\theta_1) \exp(i\mu\phi_1) \ . \tag{13}$$

Using this expression for ψ, we obtain, in view of formula (5), for the last term of the right member of equation (10),

$$\nabla\psi \times \overline{r}_1 = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} A(\mu,\nu)\, \overline{m}_{\mu\nu} , \tag{14}$$

where $\overline{m}_{\mu\nu}$ is expressed in terms of functions of the second coordinate system. By using the orthogonal properties of the angular functions

and of the vector wave functions (appendix D), it can be shown that the other vector quantities in equation (10) are

$$\nabla \psi \times i_x = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} (a'_{\mu\nu} \bar{m}_{\mu\nu} + b'_{\mu\nu} \bar{n}_{\mu\nu})$$

$$\nabla \psi \times i_y = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} (a''_{\mu\nu} \bar{m}_{\mu\nu} + b''_{\mu\nu} \bar{n}_{\mu\nu})$$

$$\nabla \psi \times i_z = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} (a'''_{\mu\nu} \bar{m}_{\mu\nu} + b'''_{\mu\nu} \bar{n}_{\mu\nu}) , \tag{15}$$

where

$$a'_{\mu\nu} = \frac{\nu}{2\nu(\nu+1)} \left\{ \frac{\nu+1}{2\nu-1} \left[A(\mu-1,\nu-1) - (\nu-\mu)(\nu-\mu-1)A(\mu+1,\nu-1) \right] + \frac{\nu}{2\nu+3} \left[(\nu+\mu+2)(\nu+\mu+1)A(\mu+1,\nu+1) - A(\mu-1,\nu+1) \right] \right\}$$

$$a''_{\mu\nu} = \frac{-ik}{2\nu(\nu+1)} \left\{ \frac{\nu+1}{2\nu-1} \left[A(\mu-1,\nu-1) + (\nu-\mu)(\nu-\mu-1)A(\mu+1,\nu-1) \right] - \frac{\nu}{2\nu+3} \left[(\nu+\mu+2)(\nu+\mu+1)A(\mu+1,\nu+1) + A(\mu-1,\nu+1) \right] \right\} \tag{16}$$

$$a'''_{\mu\nu} = \frac{k}{\nu(\nu+1)} \left\{ \frac{(\nu+1)(\nu-\mu)}{2\nu-1} A(\mu,\nu-1) + \frac{\nu(\nu+\mu+1)}{2\nu+3} A(\mu,\nu+1) \right\} ,$$

and

$$b'_{\mu\nu} = \frac{-ik}{2\nu(\nu+1)} \left[(\nu-\mu)(\nu+\mu+1) \quad A(\mu+1,\nu) + A(\mu-1,\nu) \right]$$

$$b''_{\mu\nu} = \frac{k}{2\nu(\nu+1)} \left[(\nu-\mu)(\nu+\mu+1) \quad A(\mu+1,\nu) - A(\mu-1,\nu) \right]$$

$$b'''_{\mu\nu} = \frac{ik\mu}{\nu(\nu+1)} A(\mu,\nu) . \tag{17}$$

The vector wave functions $\bar{m}_{\mu\nu}$ and $\bar{n}_{\mu\nu}$ are expressed in terms of functions of the second coordinate system.

4. REDUCTION OF TRANSLATIONAL FORMULAS

In reducing formula (10), we may, by using equation (15), set

$$A_{\mu\nu} = \sin\theta_o \cos\phi_o \, a'_{\mu\nu} + \sin\theta_o \sin\phi_o \, a''_{\mu\nu} + \cos\theta_o \, a'''_{\mu\nu} \;. \quad (18)$$

Using equations (12) and (16) with equation (18), we can write

$$A_{\mu\nu} = \left\{(-1)^{\mu} i^{\nu-n-1} k \exp\left[i(m-\mu)\phi_o\right] / \left[2\nu(\nu+1)\right]\right\} \sum_p a_p P_p^{m-\mu}(\cos\theta_o) \;, \quad (19)$$

where

$$a_p = -\begin{bmatrix} (p+m-\mu+2)(p+m-\mu+1) \quad a(m,n|-\mu+1,\nu-1|\; p+1) \\ +(\nu-\mu)(\nu-\mu-1) \quad a(m,n|-\mu-1,\nu-1|\; p+1) \\ -2(\nu-\mu)(p+m-\mu+1) \quad a(m,n|-\mu,\nu-1|\; p+1) \end{bmatrix} (\nu+1) i^{p+1} z_{p+1}(ka)/(2p+3)$$

$$-\begin{bmatrix} (p+m-\mu+2)(p+m-\mu+1) \quad a(m,n|-\mu+1,\nu+1|\; p+1) \\ +(\nu+\mu+2)(\nu+\mu+1) \quad a(m,n|-\mu-1,\nu+1|\; p+1) \\ +2(\nu+\mu+1)(p+m-\mu+1) \quad a(m,n|-\mu,\nu+1|\; p+1) \end{bmatrix} \nu\, i^{p+1} z_{p+1}(ka)/(2p+3)$$

$$+\begin{bmatrix} (p-m+\mu-1)(p-m+\mu) \quad a(m,n|-\mu+1,\nu-1|\; p-1) \\ +(\nu-\mu)(\nu-\mu-1) \quad a(m,n|-\mu-1,\nu-1|\; p-1) \\ +2(\nu-\mu)(p-m+\mu) \quad a(m,n|-\mu,\nu-1|\; p-1) \end{bmatrix} (\nu+1) i^{p-1} z_{p-1}(ka)/(2p-1)$$

$$
+\begin{bmatrix} (p-m+\mu-1)(p-m+\mu) & a(m,n|-\mu+1,\nu+1|\ p-1) \\ +(\nu+\mu+2)(\nu-\mu+1) & a(m,n|-\mu-1,\nu+1|\ p-1) \\ -2(\nu+\mu+1)(p-m+\mu) & a(m,n|-\mu,\nu+1|\ p-1) \end{bmatrix} \nu i^{p-1} z_{p-1}(ka)/(2p-1) \quad . \tag{20}
$$

If we use the recursion formulas (C-10) through (C-13), we can write

$$
a_p = \left\{ \begin{array}{l} \left[(\nu+1)(n-\nu+p+1)(n+\nu-p)-\nu(\nu-n+p+1)(n+\nu+p+2)\right] z_{p+1}(ka) \\ +\left[\nu(n+\nu-p+1)(n-\nu+p)-(\nu+1)(n+\nu+p+1)(\nu-n+p)\right] z_{p-1}(ka) \end{array} \right\}
$$

$$
i^{p+1}\ a(m,n|-\mu,\nu|p)/(2p+1) \quad . \tag{21}
$$

It can be shown that the coefficients of the radial functions $z_p(ka)$ are equal. Making use of the appropriate relation between the radial functions, we obtain

$$
a_p = \left[(\nu+1)(n-\nu+p+1)(n+\nu-p)-\nu(\nu-n+p+1)(n+\nu+p+2)\right]
$$

$$
i^{p+1}\quad a(m,n|-\mu,\nu|p)z_p(ka)/ka \quad . \tag{22}
$$

Thus

$$
A_{u\nu} = \left\{(-1)^{\mu}\ i^{\nu-n} \exp\left[i(m-\mu)\phi_o\right] / \left[2\nu(\nu+1)a\right]\right\} \sum_p a'_p\ P_p^{m-\mu}(\cos\theta_o) z_p(ka), \tag{23}
$$

where

$$
a'_p = \left[(\nu+1)(n-\nu+p+1)(n+\nu-p)-\nu(\nu-n+p+1)(n+\nu+p+2)\right] i^p a(m,n|-\mu,\nu|p). \tag{24}
$$

If we denote the coefficients of the vector wave function $\bar{m}_{u\nu}$ by $A^{mn}_{\mu\nu}$, then from equations (14), (15), and (20), we obtain

$$
A^{mn}_{\mu\nu} = a\ A_{\mu\nu} + A(\mu,\nu) \quad . \tag{25}
$$

Using the values of $A(\mu,\nu)$ from equation (12) and that of $A_{\mu\nu}$ from equation (23), we obtain

$$A_{\mu\nu}^{mn} = (-1)^{\mu} \sum_{p} a(m,n|-\mu,\nu|p)\, a(n,\nu,p)\, z_p(ka)\, P_p^{m-\mu}(\cos\theta_o)\, \exp\,[i(m-\mu)\phi_o] \tag{26}$$

where

$$a(n,\nu,p) = i^{\nu+p-n}\left[2\nu(\nu+1)(2\nu+1)+(\nu+1)(n-\nu+p+1)(n+\nu-p)\right.$$
$$\left.-\nu(\nu-n+p+1)(n+\nu+p+2)\right] / \left[2\nu(\nu+1)\right] . \tag{27}$$

For the coefficient of $\bar{n}_{\mu\nu}$ we may set

$$B_{\mu\nu}^{mn} = a(\sin\theta_o\cos\phi_o\, b'_{\mu\nu} + \sin\theta_o\sin\phi_o\, b''_{\mu\nu} + \cos\theta_o\, b'''_{\mu\nu}) \tag{28}$$

Using equations (12) and (17), we can write

$$B_{\mu\nu}^{mn} = \left\{(-1)^{\mu+1}\, i^{\nu-n+1}\, ka(2\nu+1)\, \exp\left[i(m-\mu)\phi_o\right] / \left[2\nu(\nu+1)\right]\right\}$$
$$\sum_{p} b_p\, P_p^{m-\mu}(\cos\theta_o) , \tag{29}$$

where

$$b_p = \begin{bmatrix} (p+m-\mu+1)(p+m-\mu+2)a(m,n|-\mu+1,\nu|p+1) \\ -(\nu-\mu)(\nu+\mu+1)a(m,n|-\mu-1,\nu|p+1) \\ +2\mu(p+m-\mu+1)a(n,n|-\mu,\nu|p+1) \end{bmatrix} i^{p+1}\, z_{p+1}(ka)/(2p+3)$$
$$+ \begin{bmatrix} 2\mu(p-m+\mu)\quad a(m,n|-\mu,\nu|p-1 \\ +(\nu-\mu)(\nu+\mu+1)\quad a(m,n|-\mu-1,\nu|p-1) \\ -(p-m+\mu-1)(p-m+\mu)\quad a(m,n|-\mu+1,\nu|p-1) \end{bmatrix} i^{p-1}\, z_{p-1}(ka)/(2p-1). \tag{30}$$

By using equations (C-14) and (C-7) and the appropriate relation for the radial functions, we obtain

$$b_p = -\left[(n+\nu+p+1)(\nu-n+p)(n-\nu+p)(n+\nu-p+1)\right]^{\frac{1}{2}} a(m,n|\mu,\nu|p,p-1)$$
$$i^{p+1} z_p(ka)/ka. \tag{31}$$

Substituting this value of b_p in equation (29), we obtain

$$B_{\mu\nu}^{mn} = (-1)^{\mu+1} \sum_p a(m,n|-\mu,\nu|p,p-1) \quad b(n,\nu,p) z_p(ka) \; P_p^{m-\mu}(\cos\theta_o) \exp[i(m-\mu)\phi_o], \tag{32}$$

where

$$b(n,\nu,p) = i^{\nu+p-n} \; [(n+\nu+p+1)(\nu-n+p)(n-\nu+p)(n+\nu-p+1)]^{\frac{1}{2}}(2\nu+1)/[2\nu(\nu+1)]. \tag{33}$$

5. ADDITION THEOREMS

In summary, we have the addition theorems:

Theorem I: $$\bar{m}_{mn} = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} (A_{\mu\nu}^{mn} \bar{m}_{\mu\nu} + B_{\mu\nu}^{mn} \bar{n}_{\mu\nu})$$

Theorem II: $$\bar{n}_{mn} = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} (A_{\mu\nu}^{mn} \bar{n}_{\mu\nu} + B_{\mu\nu}^{mn} \bar{m}_{\mu\nu})$$

where

$$A_{\mu\nu}^{mn} = (-1)^{\mu} \sum_p a(m,n|-\mu,\nu|p) \quad a(n,\nu,p) z_p(ka) \; P_p^{m-\mu}(\cos\theta_o) \exp[i(m-\mu)\phi_o], \tag{34}$$

$$B_{\mu\nu}^{mn} = (-1)^{\mu+1} \sum_p a(m,n|-\mu,\nu|p,p-1) \quad b(n,\nu,p) z_p(ka) \; P_p^{m-\mu}(\cos\theta_o) \exp[i(m-\mu)\phi_o],$$

$$a(n,\nu,p) = i^{\nu+p-n} \; [2\nu(\nu+1)(2\nu+1)+(\nu+1)(n-\nu+p+1)(n+\nu-p)$$

$$-\nu(\nu-n+p+1)(n+\nu+p+2)]/[2\nu(\nu+1)],$$

and

$$b(n,\nu,p) = i^{\nu+p-n} \; [(n+\nu+p+1)(\nu-n+p)(n-\nu+p)(n+\nu-p+1)]^{\frac{1}{2}} \; (2\nu+1)/[2\nu(\nu+1)]. \tag{35}$$

APPENDIX A

COEFFICIENTS IN EXPANSION OF $P_n^m P_\nu^\mu$

In the addition theorem for the scalar spherical wave function (appendix B), there occurs the produce $P_n^m P_\nu^\mu$ of two associated Legendre functions in which the argument is cos θ. For purposes of convenience it is desirable to express this product as a sum of terms in $P_{p\nu}^{m''}$, where $m'' = m + \mu$ and p is the summation index. Formally, the expansion is

$$P_n^m P_\nu^\mu = \sum_p a(m,n|\mu,\nu|p)\; P_p^{m+\mu} \quad . \tag{A-1}$$

The coefficients $a(m,n|\mu,\nu|p)$ can be obtained in the following way. It is known that the integral of the product of three spherical harmonics over the surface of a unit sphere is given by (ref 4)

$$\int_0^{2\pi}\int_0^{\pi} Y_{mn}\, Y_{\mu\nu}\, Y_{m'p} \sin\theta\, d\theta\, d\phi$$

$$= \left[\frac{(2n+1)(2\nu+1)(2p+1)}{4\pi}\right]^{\frac{1}{2}} \begin{pmatrix} n & \nu & p \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} n & \nu & p \\ m & \mu & m' \end{pmatrix} \quad , \tag{A-2}$$

where

$$Y_{rs}(\theta,\phi) = (-1)^s \left[\frac{(2r+1)(r-s)!}{4\pi\;(r+s)!}\right]^{\frac{1}{2}} P_r^s (\cos\theta \exp(is\phi) \quad , \tag{A-3}$$

and

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \tag{A-4}$$

is the Wigner 3-j symbol (ref 5). (An expression for the 3-j symbol is given at the end of this appendix.)

The product of two spherical harmonics of the form as given by equation (A-3) is

$$Y_{mn}(\theta,\phi)\ Y_{\mu\nu}(\theta,\phi) = \frac{(-1)^{m+\mu}}{4\pi}\left[\frac{(2n+1)(2\nu+1)(n-m)!(\nu-\mu)!}{(n+m)!\ (\nu+\mu)!}\right]^{\frac{1}{2}} P_n^m (\cos\theta)\ P_\nu^\mu (\cos\theta)$$

$$\exp\left[i(m+\mu)\phi\right] . \qquad \text{(A-5)}$$

Formally, in view of expression (A-1), let us write equation (A-5) as

$$Y_{mn}(\theta,\phi)\ Y_{\mu\nu}(\theta,\phi) = \frac{(-1)^{m+\mu}}{4\pi}\left[\frac{(2n+1)(2\nu+1)(n-m)!(\nu-\mu)!}{(n+m)!\ (\nu+\mu)!}\right]^{\frac{1}{2}} \exp\left[i(m+\mu)\phi\right]$$

$$\cdot \sum_p a(m,n|\mu,\nu|p)\ P_p^{m+\mu} . \qquad \text{(A-6)}$$

If we take, where $m'=-m-\mu$,

$$Y_{pm'}(\theta,\phi) = (-1)^{-(m+\mu)}\left[\frac{(2p+1)(p+m+\mu)!}{4\pi\ (p-m-\mu)!}\right]^{\frac{1}{2}} P_p^{-m-\mu}(\cos\theta)\ \exp\left[i(m+\mu)\phi\right] , \qquad \text{(A-7)}$$

Noting that

$$P_p^{-m-\mu}(\cos\theta) = (-1)^{m+\mu}\ \frac{(p-m-\mu)!}{(p+m+\mu)!}\ P_p^{m+\mu}\ (\cos\theta) \quad , \qquad \text{(A-8)}$$

and the product of two spherical harmonics as given by expression (A-6), we find that the left-hand member of equation (A-2), which we denote by "A", is given by

$$A= (-1)^{m+\mu}\left[\frac{(2n+1)(2\nu+1)(n-m)!(\nu-\mu)!(p+m+\mu)!}{4\pi\ (n+m)!\ (\nu+\mu)!(p-m-\mu)!(2p+1)}\right] a(m,n|\mu,\nu|p) . \qquad \text{(A-9)}$$

Upon setting this equal to the right-hand member of equation (A-2), we find for the coefficients:

$$a(m,n|\mu,\nu|p)= (-1)^{m+\mu}\ (2p+1)\ \left[\frac{(n+m)!(\nu+\mu)!(p-m-\mu)!}{(n-m)!(\nu-\mu)!(p+m+\mu)!}\right]^{\frac{1}{2}} \cdot \qquad \text{(A-10)}$$

$$\cdot \begin{pmatrix} n & \nu & p \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} n & \nu & p \\ m & \mu & -m-\mu \end{pmatrix} .$$

The conditions that the indices must satisfy are

$$- p \leq m+\mu \leq p \tag{A-11}$$

$$|n-\nu| \leq p \leq n+\nu \tag{A-12}$$

and

$$(n+\nu+p) \text{ even.} \tag{A-13}$$

If $(m+\mu)$ does not lie in the range as given by expression (A-11), then $P_p^{m+\mu}$ vanishes; also, if p does not lie in the range as given by expression (A-12), then the coefficients $a(m,n|\mu,\nu|p)$ vanish. Furthermore, if condition (A-13) is not satisfied, the Wigner 3-j symbol

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix}$$

vanishes. The Wigner 3-j symbol is defined as follows;

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1-j_2-m_3}(2j_3+1)^{-\frac{1}{2}}(j_1m_1j_2m_2|j_1j_2j_3\ -m_3)\ , \tag{A-14}$$

where

$$(j_1m_1j_2m_2|j_1j_2j\ m) = \delta(m_1+m_2,m)$$

$$\left[\frac{(2j+1)(j_1+j_2-j)!(j_1-m_1)!(j_2-m_2)!(j+m)!(j-m)!}{(j_1+j_2+j+1)!(j_1-j_2+j)!(-j_1+j_2+j)!(j_1+m_1)!(j_2+m_2)!}\right]^{\frac{1}{2}}$$

$$\cdot \sum_s (-1)^{s+j_1-m_1} \frac{(j_1+m_1+s)!(j_2+j-m_1-s)!}{s!(j_1-m_1-s)!(j-m-s)!(j_2-j+m_1+s)!} \ . \tag{A-15}$$

APPENDIX B

ADDITION THEOREMS FOR SCALAR SPHERICAL WAVE FUNCTIONS

In this appendix, the addition theorems for the scalar wave functions will be listed, with appropriate limits, since they have been already derived (ref 6). The addition theorems given here are for the case of coordinate translation illustrated in figure 1. The more correct forms for the theorems (ref 7) are

$$z_n(kr)\ P_n^m(\cos\theta)\ \exp(im\phi) = \sum_{\nu=0}^{\infty}\ \sum_{\mu=-\nu}^{\nu}\ \sum_{p} (-1)^{\mu}\ i^{\nu+p-n} a(m,n|-\mu,\nu|p)$$

$$\cdot\ j_\nu(kr_1)\ z_p(ka)\ P_\nu^\mu(\cos\theta_1)\ P_p^{m-\mu}(\cos\theta_o)\ \exp[i(m-\mu)\phi_1] \exp(i\mu\phi_1)$$

$$r_1 \leq a \qquad \text{(B-1)}$$

$$= \sum_{\nu=0}^{\infty}\ \sum_{\mu=-\nu}^{\nu}\ \sum_{p} (-1)^{\mu}\ i^{\nu+p-n}\ a(m,n|-\mu,\nu|p)$$

$$\cdot\ j_\nu(ka)\ z_p(kr_1)\ P_\nu^\mu(\cos\theta_o)\ P_p^{m-\mu}(\cos\theta_1)\ \exp[i(m-\mu)\phi_o]\ \exp(i\mu\phi_o)$$

$$r_1 \geq a\ . \qquad \text{(B-2)}$$

The symbol $z_n(kr)$ stands for the spherical Bessel, Neumann, or Hankel (first or second kind) function. It can be shown that either (B-1) or (B-2) may be used for $z_n(kr) = j_n(kr)$ without restriction on the relative size of r_1 and a.

APPENDIX C

RELATIONSHIPS INVOLVING THE COEFFICIENTS IN THE EXPANSION OF $a(m,n|\mu,\nu|p)$

For the reduction of the coefficients involved in the addition theorems, it is necessary to obtain formulas simplifying various groupings of the $a(m,n|\mu,\nu|p)$'s. It so happens that some of the symbols and formulas of group theory are quite fitted for this purpose.

From group theory there is the relation between the 3-j symbol and 6-j symbol (ref 8):

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{Bmatrix} j_1 & j_2 & j_3 \\ \ell_1 & \ell_2 & \ell_3 \end{Bmatrix} = \sum_{\mu_1 \mu_2 \mu_3} (-1)^{\ell_1+\ell_2+\ell_3+\mu_1+\mu_2+\mu_3}$$

$$\begin{pmatrix} j_1 & \ell_2 & \ell_3 \\ m_1 & \mu_2 & -\mu_3 \end{pmatrix} \begin{pmatrix} \ell_1 & j_2 & \ell_3 \\ -\mu_1 & m_2 & \mu_3 \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ \mu_1 & -\mu_2 & m_3 \end{pmatrix},$$

where

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \text{ and } \begin{Bmatrix} j_1 & j_2 & j_3 \\ \ell_1 & \ell_2 & \ell_3 \end{Bmatrix} \tag{C-1}$$

are respectively, 3-j and 6-j symbols. If the ℓ's are given special values, then the following recursion formulas are obtained:

FORMULA I (ref 9)

$$\ell_1 = 1, \ \ell_2 = j_3 - 1, \ \ell_3 = j_2$$

$$[(J+1)(J-2j_1)(J-2j_2)(J-2j_3+1)]^{\frac{1}{2}} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

$$= -2m_2 \, [(j_3+m_3)(j_3-m_3)]^{\frac{1}{2}} \begin{pmatrix} j_1 & j_2 & j_3-1 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

$$-[(j_2+m_2)(j_2-m_2+1)(j_3-m_3)(j_3-m_3-1)]^{\frac{1}{2}}\begin{pmatrix} j_1 & j_2 & j_3-1 \\ m_1 & m_2-1 & m_3+1 \end{pmatrix}$$

$$+[(j_2-m_2)(j_2+m_2+1)(j_3+m_3)(j_3+m_3-1)]^{\frac{1}{2}}\begin{pmatrix} j_1 & j_2 & j_3-1 \\ m_1 & m_2+1 & m_3-1 \end{pmatrix} \qquad \text{(C-2)}$$

FORMULA II

$$\ell_1 = 1,\ \ell_2=j_3+1,\ \ell_3=j_2$$

$$[(J-2j_1+1)(J-2j_2+1)(J-2j_3)(J+2)]^{\frac{1}{2}}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

$$= -2m_2\ [(j_3+m_3+1)(j_3-m_3+1)]^{\frac{1}{2}}\begin{pmatrix} j_1 & j_2 & j_3+1 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

$$+ [(j_2+m_2)(j_2-m_2+1)(j_3+m_3+1)(j_3+m_3+2)]^{\frac{1}{2}}\begin{pmatrix} j_1 & j_2 & j_3+1 \\ m_1 & m_2-1 & m_3+1 \end{pmatrix}$$

$$- [(j_2-m_2)(j_2+m_2+1)(j_3-m_3+1)(j_3-m_3+2)]^{\frac{1}{2}}\begin{pmatrix} j_1 & j_2 & j_3+1 \\ m_1 & m_2+1 & m_3-1 \end{pmatrix} . \qquad \text{(C-3)}$$

FORMULA III

$$\ell_1 = 1,\ \ell_2 = j_3-1,\ \ell_3 = j_2+1$$

$$[(J-2j_2)(J-2j_2-1)(J-2j_3+1)(J-2j_3+2)]^{\frac{1}{2}} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

$$= 2[(j_3+m_3)(j_3-m_3)(j_2+m_2+1)(j_2-m_2+1)]^{\frac{1}{2}} \begin{pmatrix} j_1 & j_2+1 & j_3-1 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

$$+ [(j_3-m_3)(j_3-m_3-1)(j_2-m_2+1)(j_2-m_2+2)]^{\frac{1}{2}} \begin{pmatrix} j_1 & j_2+1 & j_3-1 \\ m_1 & m_2-1 & m_3+1 \end{pmatrix}$$

$$+[(j_3+m_3)(j_3+m_3-1)(j_2+m_2+1)(j_2+m_2+2)]^{\frac{1}{2}} \begin{pmatrix} j_1 & j_2+1 & j_3-1 \\ m_1 & m_2+1 & m_3-1 \end{pmatrix} \quad . \quad (C-4)$$

FORMULA IV

$$\ell_1 = 1,\ \ell_2 = j_3-1,\ \ell_3 = j_2-1$$

$$[J(J+1)(J-2j_1)(J-2j_1-1)]^{\frac{1}{2}} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

$$= -2[(j_2+m_2)(j_2-m_2)(j_3+m_3)(j_3-m_3)]^{\frac{1}{2}} \begin{pmatrix} j_1 & j_2-1 & j_3-1 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

$$+ [(j_2+m_2)(j_2+m_2-1)(j_3-m_3)(j_3-m_3-1)]^{\frac{1}{2}} \begin{pmatrix} j_1 & j_2-1 & j_3-1 \\ m_1 & m_2-1 & m_3+1 \end{pmatrix}$$

$$+[(j_2-m_2)(j_2-m_2-1)(j_3+m_3)(j_3+m_3-1)]^{\frac{1}{2}} \begin{pmatrix} j_1 & j_2-1 & j_3-1 \\ m_1 & m_2+1 & m_3-1 \end{pmatrix} . \qquad \text{(C-5)}$$

FORMULA V

$$\ell_1 = 1, \quad \ell_2 = j_3+1, \quad \ell_3 = j_2+1$$

$$[(J+3)(J+2)(J-2j_1+2)(J-2j_1+1)]^{\frac{1}{2}} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

$$= -2[(j_2-m_2+1)(j_2+m_2+1)(j_3-m_3+1)(j_3+m_3+1)]^{\frac{1}{2}} \begin{pmatrix} j_1 & j_2+1 & j_3+1 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

$$+[(j_2-m_2+1)(j_2-m_2+2)(j_3+m_3+1)(j_3+m_3+2)]^{\frac{1}{2}} \begin{pmatrix} j_1 & j_2+1 & j_3+1 \\ m_1 & m_2-1 & m_3+1 \end{pmatrix}$$

$$+[(j_2+m_2+1)(j_2+m_2+2)(j_3-m_3+1)(j_3-m_3+2)]^{\frac{1}{2}} \begin{pmatrix} j_1 & j_2+1 & j_3+1 \\ m_1 & m_2+1 & m_3-1 \end{pmatrix} . \qquad \text{(C-6)}$$

In all formulas, $J=j_1 + j_2 + j_3$.

If in Formula III, columns (2) and (3) are interchanged, then it will be observed that with formulas IV and V there are four cases in which the sums of the j's in all 3-j symbols are even or odd simultaneously. With

formulas I and II this does not occur; here when those in the right members are odd, the sums of the j's in the left members are even and conversely. These facts will show in the relationships between the $a(m,n|\mu,\nu|p)$'s based upon these formulas, that there will be four pure recursion formulas with the other two involving "hybrid" terms which are closely related to the $a(m,n|\mu,\nu|p)$'s in form.

By giving the j's and the m's special values and multiplying by the appropriate factor, as is evidenced from the form of the $a(m,n|\mu,\nu|p)$ in Appendix A, we obtain the following relationships:

Case I

In formula I, let

$$j_1 = n, \quad j_2 = \nu, \; j_3 = p$$

$$m_1 = m, \quad m_2 = \mu, \; m_3 = -m-\mu,$$

then

$$(2p-1)[(n+\nu+p+1)(\nu-n+p)(n-\nu+p)(n+\nu-p+1)]^{\frac{1}{2}} a(m,n|\mu,\nu|p,p-1)$$

$$= (2p+1)[(\nu+\mu)(\nu-\mu+1)a(m,n|\mu-1,\nu|p-1)$$

$$-(p-m-\mu)(p-m-\mu-1)a(m,n|\mu+1,\nu|p-1)$$

$$-2\mu\;(p-m-\mu)a(m,n|\mu,\nu|p-1)\,] \,. \qquad \text{(C-7)}$$

Case II

In formula II, let

$$j_1 = n, \; j_2 = \nu, \; j_3 = p$$

$$m_1 = m, \; m_2 = \mu, \; m_3 = -m-\mu,$$

then

$$(2p+3)\;[(n+\nu-p)(n-\nu+p+1)(\nu+p-n+1)(n+\nu+p+2)]^{\frac{1}{2}}\; a(m,n|\mu,\nu|p,p+1)$$

$$= (2p+1)\left[(p+m+\mu+\mu+1)(p+m+\mu+2)a(m,n|\mu+1,\nu|p+1)\right.$$

$$-(\nu+\mu)(\nu-\mu+1)a(m,n|\mu-1,\nu|p+1)$$

$$\left.-2\mu(p+m+\mu+1)a(m,n|\mu,\nu|p+1)\right] \quad , \qquad \text{(C-8)}$$

where

$$a(m,n|\mu,\nu|p,q) = (-1)^{m+\mu}\left[\frac{(n+m)!(\nu+\mu)!(p-\mu-m)!}{(n-m)!(\nu-\mu)!(p+\mu+m)!}\right](2p+1)$$

$$\cdot \begin{pmatrix} n & \nu & p \\ m & \mu & -m-\mu \end{pmatrix} \begin{pmatrix} n & \nu & q \\ 0 & 0 & 0 \end{pmatrix} . \qquad \text{(C-9)}$$

<u>Case III</u>

In formula III, let

$$j_1 = n, \quad j_2 = \nu-1, \quad j_3 = p$$

$$m_1 = m, \quad m_2 = \mu \quad , \quad m_3 = -m-\mu,$$

then

$$(2p-1)(n+\nu-p)(n-\nu+p+1)a(m,n|\mu,\nu-1|p)$$

$$= (2p+1)\left[2(\nu-\mu)(p-m-\mu)a(m,n|\mu,\nu|p-1)\right.$$

$$-(\nu-\mu)(\nu-\mu+1)a(m,n|\mu-1,\nu|p-1)$$

$$\left.-(p-m-\mu-1)(p-m-\mu)a(m,n|\mu+1,\nu|p-1)\right] \qquad \text{(C-10)}$$

Case IV (ref 10)

In formula III, interchange columns (2) and (3) and let

$$j_1 = n, \quad j_3 = \nu + 1, \quad j_2 = p$$

$$m_1 = m, \quad m_3 = \mu, \quad m_2 = -m-\mu,$$

then

$$(2p+3)(n-\nu+p)(n+\nu-p+1)a(m,n|\mu,\nu+1|p)$$

$$= (2p+1)[\, -(p+m+\mu+1)(p+m+\mu+2)a(m,n|\mu+1,\nu|p+1)$$

$$+2(\nu+\mu+1)(p+m+\mu+1)a(m,n|\mu,\nu|p+1)$$

$$-(\nu+\mu)(\nu+\mu+1)a(m,n|\mu-1,\nu|p+1)] \;. \tag{C-11}$$

Case V (ref 10)

In formula IV, let

$$j_1 = n, \quad j_2 = \nu + 1, \quad j_3 = p$$

$$m_1 = m, \quad m_2 = \mu, \quad m_3 = -m-\mu,$$

then

$$(2p-1)(n+\nu+p+2)(\nu-n+p+1)a(m,n|\mu,\nu+1|p)$$

$$= (2p+1)\,[\,(p-m-\mu-1)(p-m-\mu)a(m,n|\mu+1,\nu|p-1)$$

$$+2(\nu+\mu+1)(p-m-\mu)a(m,n|\mu,\nu|p-1)$$

$$+(\nu+\mu)(\nu+\mu+1)a(m,n|\mu-1,\nu|p-1)\,] \;. \tag{C-12}$$

Case VI

In formula V, let

$$j_1 = n, \quad j_2 = \nu - 1, \quad j_3 = p$$

$$m_1 = m, \quad m_2 = \nu\;, \quad m_3 = -m-\mu,$$

then

$$(2p+3)(\nu-n+p)(n+\nu+p+1)a(m,n|\mu,\nu-1|p)$$

$$=(2p+1)\left[\ (p+m+\mu+2)(p+m+\mu+1)a(m,n|\mu+1,\nu|p+1)\right.$$

$$+(\nu-\mu)(\nu-\mu+1)a(m,n|\mu-1,\nu|p+1)$$

$$\left.+2(\nu-\mu)(p+m+\mu+1)a(m,n|\mu,\nu|p+1)\right] \ . \qquad \text{(C-13)}$$

In addition to the above relationships for the $a(m,n|\mu,\nu|p)$'s, there can be derived numerous recursion formulas based upon various recursion relations for the associated Legendre functions. By using the relationship for associated Legendre functions,

$$\left[(1-n^2)^{\frac{1}{2}}\ P_\nu^{\mu+1} + (\nu+\mu)(\nu-\mu+1)\ P_\nu^{\mu-1} - 2\mu n\ P_\nu^\mu\right]\ P_n^m = 0,$$

it can be shown that

$$(2p-1)\left[\ (p+m+\mu+1)(p+m+\mu+2)a(m,n|\mu+1,\nu|p+1)\right.$$

$$-(\nu+\mu)(\nu-\mu+1)a(m,n|\mu-1,\nu|p+1)$$

$$\left.-2\mu(p+m+\mu+1)a(m,n|\mu,\nu|p+1)\right]$$

$$=(2p+3)\left[\ (p-m-\mu)(p-m-\mu-1)a(m,n|\mu+1,\nu|p-1)\right.$$

$$-(\nu+\mu)(\nu-\mu+1)a(m,n|\mu-1,\ \nu|p-1)$$

$$\left.+2\mu(p-m-\mu)a(m,n|\mu,\nu|p-1)\ \right] \ . \qquad \text{(C-14)}$$

Certain three-term recursion formulas are also readily obtained. From the relation

$$\frac{d}{d\theta}\left(P_n^m\ P_\nu^\mu\right) = P_n^m\ \frac{dP_\nu^\mu}{d\theta} + P_\nu^\mu\ \frac{dP_n^m}{d\theta}\ , \qquad \text{(C-15)}$$

and the recursion formulas

$$\frac{dP_{n'}^{m'}}{d\theta} = \frac{m'n}{(1-\eta^2)^{\frac{1}{2}}}\, P_{n'}^{m'} - P_{n'}^{m'+1} \qquad \text{(C-16)}$$

$$\frac{dP_{n'}^{m'}}{d\theta} = \frac{-m'\eta}{(1-\eta^2)} P_{n'}^{m'} + (n'-m'+1)(n'+m') P_{n'}^{m'-1} , \tag{C-17}$$

one obtains respectively

$$a(m,n|\mu,\nu|p)=a(m+1,n|\mu,\nu|p)+a(m,n|\mu+1,\nu|p) \tag{C-18}$$

and

$$(p-m-\mu+1)(p+m+\mu)a(m,n|\mu,\nu|p)=(\nu+\mu)(\nu-\mu+1)a(m,n|\mu-1,\nu|p)$$

$$+(n+m)(n-m+1)a(m-1,n|\mu,\nu|p). \tag{C-19}$$

From (C-18) and (C-19) there can be obtained the formulas

$$[(p+m+\mu)(p-m-\mu+1)+(\nu-\mu)(\nu+\mu+1)-(n+m)(n-m+1)]\ a(m,n|\mu,\nu|p)$$

$$= (\nu+\mu)(\nu-\mu+1)a(m,n|\mu-1,\nu|p)$$

$$+(p-m-\mu)(p+m+\mu+1)a(m,n|\mu+1,\nu|p) \tag{C-20}$$

and

$$[(p+m+\mu)(p-m-\mu+1)+(n-m)(n+m+1)-(\nu+\mu)(\nu-\mu+1)]\ a(m,n|\mu,\nu|p)$$

$$= (n+m)(n-m+1)a(m-1,n|\mu,\nu|p)$$

$$+(p-m-\mu)(p+m+\mu+1)a(m+1,n|\mu,\nu|p) \ . \tag{C-21}$$

APPENDIX D

ORTHOGONAL RELATIONS

In the expansion of vector qualities in terms of vector wave functions, various orthogonal relations are used. For the divergenceless vector wave functions we have the following relations:

$$\int_{-1}^{+1}\int_{0}^{2\pi} \bar{m}_{mn} \cdot \bar{m}^*_{mn} \, d\phi d\eta = \frac{4\pi\, n(n+1)}{2n+1} \frac{(n+m)!}{(n-m)!} [z_n(kr)]^2$$

$$\int_{-1}^{+1}\int_{0}^{2\pi} \bar{n}_{mn} \cdot \bar{n}^*_{mn} \, d\phi d\eta = \frac{4\pi}{(2n+1)^2} \frac{(n+m)!}{(n-m)!} n(n+1)$$

$$\cdot \left\{ (n+1 \, [z_{n-1}(kr)]^2 + n \, [z_{n+1}(kr)]^2 \right\}$$

$$\int_{-1}^{+1}\int_{0}^{2\pi} \bar{m}_{mn} \cdot \bar{n}^*_{m'n'} \, d\phi d\eta = 0. \qquad \text{(D-1)}$$

For the associated Legendre functions the following relations are useful:

$$\int_{-1}^{+1} (1-\eta^2)^{\frac{1}{2}} \left[\frac{\partial P^{m+1}_{n'}}{\partial\theta} \frac{\partial P^m_n}{\partial\theta} + \frac{m(m+1)}{1-\eta^2} P^{m+1}_{n'} P^m_n \right] d\eta = \begin{cases} \dfrac{-2(n^2-1)}{(2n-1)(2n+1)} \dfrac{(n+m)!}{(n-m-2)!} & n'=n-1 \\[2ex] \dfrac{2[(n+1)^2-1]}{(2n+1)(2n+3)} \dfrac{(n+m+2)!}{(n-m)!} & n'=n+1 \end{cases}$$

$$\int_{-1}^{+1} \eta \left[\frac{\partial P_n^m}{\partial \theta} \frac{\partial P_{n'}^m}{\partial \theta} + \frac{m^2}{1-\eta^2} P_n^m \; P_{n'}^m \right] d\eta = \begin{cases} \dfrac{2[(n+1)^2-1]}{(2n+1)(2n+3)} \dfrac{(n+m+1)!}{(n-m)!} & n'=n+1 \\[2ex] \dfrac{2(n^2-1)}{(2n-1)(2n+1)} \dfrac{(n+m)!}{(n-m).} & n'=n-1 \end{cases} \tag{D-2}$$

$$\int_{-1}^{+1} P_{n'}^{m-1} \left[\eta \frac{\partial P_n^m}{\partial \theta} + \frac{mP_n^m}{(1-\eta^2)^{\frac{1}{2}}} \right] d\eta = \begin{cases} \dfrac{2(n+1)}{(2n-1)(2n+1} \dfrac{(n+m)!}{(n-m)!} & n'=n-1 \\[2ex] \dfrac{2n}{(2n+1)(2n+3)} \dfrac{(n+m)!}{(n-m).} & n'=n+1 \end{cases}$$

$$\int_{-1}^{+1} P_{n'}^{m+1} \left[\eta \frac{\partial P_n^m}{\partial \theta} - \frac{mP_n^m}{(1-\eta^2)^{\frac{1}{2}}} \right] d\eta = \begin{cases} \dfrac{-2(n+1)}{(2n-1)(2n+1)} \dfrac{(n+m)!}{(n-m-2)!} & n'=n-1 \\[2ex] \dfrac{-2n}{(2n+1)(2n+3)} \dfrac{(n+m+2)!}{(n-m)!} & n'=n+1 \end{cases} .$$

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Translational addition theorems for spherical wave functions have been derived in a reduced form. The reduction has been accomplished by the use of formulas relating the coefficients that arise in expansion of the product of two associated Legendre functions. These addition theorems should be useful in those cases in which spherical vector wave functions are used where the distances of bodies and sources are separated by the order of a few wavelengths.

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